Electrostatic modes in dusty plasmas with continuous size distributions

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When the dust plasma frequency, and hence the dust-acoustic velocity is computed for a dusty plasma containing charged grains with individual identities, three possibilities occur in a natural way. One form is based on the average over all dust grains of the ratio of the square of charge to mass, whereas a second one uses the average charge and the average mass. The difference between the two gives rise to a dust distribution mode. A third option is to describe dust grains of similar composition by a monodisperse model based on an average radius, that conserves overall charge density. The dust plasma frequency thus obtained is intermediate between those from the two other definitions, indicating that the use of a monodisperse description at this average size underestimates the mass effects of the distribution. These results are applied to power-law size distributions observed in planetary rings.

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I. INTRODUCTION

Wave processes in dusty plasmas give rise to interesting difficulties and complications that do not occur in standard plasma theory, and typically include new low-frequency modes such as the well-known dust-acoustic mode [1,2]. Background information about dusty plasmas, mixtures of electrons, ions and charged dust grains, and about their characteristic eigenmodes can be found in two recent books [3,4]. Contrary to the experiments with monodisperse dust or with grains of two sizes, dust in heliospheric and astrophysical environments comes in all sizes, in an almost continuous range going from macromolecules to rock fragments. Comparatively less research has been done about such distributions in size, mass, or charge. Models have treated the charged dust as a limited number of discrete (negative ion) species, or by using continuous distributions over a definite size range, leading to changes in dispersion laws for some of the electrostatic dusty plasma modes such as dust-acoustic waves [5–8].

To avoid difficulties associated with multiple dust grain identities, we have in a previous paper [8] studied global dust charge and mass densities and now treat continuous size distributions in detail. We recall in Sec. II elements of our earlier formalism, and give new insights into separable distributions and their consequences, when charge distributions can be considered irrespective of the momenta. The results are adapted in Sec. III to size distributions and applied to power-law size distributions observed in heliospheric plasmas. Our conclusions are summarized in Sec. IV.

II. BASIC FORMALISM

We consider a model that contains, besides the electrons and (plasma) ions, a collection of charged dust grains with differing characteristics, in contrast to the classic picture of dusty plasmas where all dust grains are assumed to have the same average mass and charge [3,4]. The grains are treated as constant point charges, since charging processes or grain interactions are not considered here. The electron and ordinary ion contributions to the electrostatic dispersion law are standard.

In most astrophysical situations, there is a whole range of sizes and masses of grains, and through the actions of the charging processes also a range of charges. All the charged dust grains (with their multiple identities) are treated together as a global dust fluid, with charge density σ_d and mass density ρ_d. We follow a simplified version [8] of the kinetic approach of Varma [9], in which the microscopic dust distribution function $f_d(\mathbf{x},\mathbf{w},t,q,m)$ incorporates the charge q and the mass m as continuous, independent phase space variables, besides $\mathbf{x}$, $\mathbf{w}$, and t. The Vlasov-type kinetic equation

$$\frac{\partial f_d}{\partial t}+\mathbf{w}\cdot\frac{\partial f_d}{\partial \mathbf{x}}-\left(\frac{q}{m}\nabla\varphi+\nabla\psi\right)\cdot\frac{\partial f_d}{\partial \mathbf{w}}=0 \tag{1}$$

includes electrostatic and self-gravitational forces through the respective potentials φ and ψ. There are no $\dot{q}$ and $\dot{m}$ contributions for dust grains with constant charges and masses, in contrast to recent attempts to describe charge fluctuations for grains that are monodisperse in mass, but polydisperse in charge, so that q but not m is treated as an additional phase space variable [10–12].

If we associate with each microscopic quantity $\mathcal{A}$ its macroscopic average $\langle\mathcal{A}\rangle$, defined as

$$\langle\mathcal{A}\rangle=\int f_d\mathcal{A}d^3\mathbf{w}dq\,dm, \tag{2}$$

the mass and charge densities are given by $\rho_d = \langle m \rangle$ and $\sigma_d = \langle q \rangle$. Since different charge-to-mass ratios are involved, two average velocities can be defined for the dust, $\mathbf{u}_d$ from following its mass motion and $\mathbf{v}_d$ from following its charge motion. This leads to two evolution equations, viz., the classic momentum equation for $\rho_d \mathbf{u}_d = \langle m\mathbf{w} \rangle$, and an equation for the evolution of the dust current $\sigma_d \mathbf{v}_d = \langle q\mathbf{w} \rangle$, written in the cold dust limit as

$$\rho_d \frac{d\mathbf{u}_d}{dt} = -\sigma_d \nabla \varphi - \rho_d \nabla \psi,$$

$$\sigma_d \frac{d\mathbf{v}_d}{dt} = -\xi_d \nabla \varphi - \sigma_d \nabla \psi. \tag{3}$$

It is necessary to include a third macroscopic density $\xi_d = \langle q^2/m \rangle$ that gives the correct definition of the dust plasma frequency for mass distributions [8], and thus has a direct impact on electrostatic modes. The set of basic equations is closed by the electrostatic and gravitational Poisson's equations.

Before proceeding further, we now briefly discuss new insights into the implications of separable distribution functions. By this we mean that the microscopic velocity properties of a given spectrum of charges and masses can be separated from the charge and mass distribution itself,

$$f_d(\mathbf{x},\mathbf{w},t,q,m) = \mathcal{F}_d(\mathbf{x},\mathbf{w},t)\mathcal{G}_d(\mathbf{x},t,q,m). \tag{4}$$

The assumption underlying such a decomposition is that there is a generic velocity distribution in phase space, that is independent of the dust charge (and mass) distributions [9,12]. This is a legitimate way of studying the charge (and mass) distributions themselves. When investigating what happens to higher q and m moments, however, the hypothesis of separability has surprising and stringent consequences, leading essentially to all charged dust having the same charge-to-mass ratio. To see this, we will separate out the averages over $\mathcal{F}_d$ and over $\mathcal{G}_d$ in an obvious notation such as $\rho_d = \langle m \rangle = \langle 1 \rangle_{\mathcal{F}} \langle m \rangle_{\mathcal{G}}$. Consequently

$$\mathbf{u}_d = \frac{\langle m\mathbf{w} \rangle}{\langle m \rangle} = \frac{\langle \mathbf{w} \rangle_{\mathcal{F}} \langle m \rangle_{\mathcal{G}}}{\langle 1 \rangle_{\mathcal{F}} \langle m \rangle_{\mathcal{G}}} = \frac{\langle \mathbf{w} \rangle_{\mathcal{F}}}{\langle 1 \rangle_{\mathcal{F}}} \tag{5}$$

is clearly independent of the charge and mass distribution itself, as expected. Analogous computations for $\mathbf{v}_d$ show that the $\langle q \rangle_{\mathcal{G}}$ contributions similarly drop out and the two macroscopic velocities coincide. This comes as no surprise, since we have assumed that the charge and mass distributions can be treated independently of the velocity distribution. In itself, this is not sufficient to prove directly that all dust grains have the same charge-to-mass ratio, although the converse obviously holds [8]. However, for $\mathbf{u}_d = \mathbf{v}_d$ (3) tells us that

$$\frac{d\mathbf{u}_d}{dt} = -\frac{\sigma_d}{\rho_d} \nabla \varphi - \nabla \psi = -\frac{\xi_d}{\sigma_d} \nabla \varphi - \nabla \psi, \tag{6}$$

necessitating $\xi_d \rho_d = \sigma_d^2$. This result is not restricted to cold dust, since one can show that it holds even with pressure terms in Eq. (3). However, $\xi_d \rho_d = \sigma_d^2$ is only possible if all dust grains have the same charge-to-mass ratio, as we now prove in general, also for dust distribution functions that are not separable. As shown in our previous paper [8] $\xi_d \rho_d - \sigma_d^2 \geqslant 0$ and the equality sign leads to

$$\int g_d(q,m) g_d(q',m') m m' \left(\frac{q}{m} - \frac{q'}{m'} \right)^2 dq\, dq'\, dm\, dm' = 0. \tag{7}$$

In the distributions g_d the integrations over microscopic velocities have been carried out and $\mathbf{x}$ and t are omitted for notational brevity. A change of integration variables to $\tau = q/m$ and $\tau' = q'/m'$ transforms this, after integrating out over m and m', into

$$\int h_d(\tau) h_d(\tau') (\tau - \tau')^2 d\tau\, d\tau' = 0. \tag{8}$$

Going through the same steps, starting from the definition of $\langle m \rangle > 0$, yields

$$\langle m \rangle = \int h_d(\tau) d\tau > 0. \tag{9}$$

Combining this with Eq. (8), one sees that $h_d(\tau)$ has a Dirac δ-distribution behavior, $h_d(\tau) \propto \delta(\tau - \tau_0)$, indicating that all charged dust particles have the same charge-to-mass ratio τ_0.

However, $\xi_d \rho_d = \sigma_d^2$ is here a consequence of the assumed separability of the original microscopic dust distribution functions f_d, so that being able to describe the dust charge (and mass) distributions irrespective of their velocities for all averages [9] is tantamount to saying that all dust grains have the same charge-to-mass ratio.

If, in addition, one considers a charge distribution for particles with the same mass, as some treatments have assumed [10,11], separability means that all particles have the same charge, and there is no longer a charge distribution One cannot really disentangle the velocity or momentum aspects from the charge and mass distributions itself, except for the lowest order average. Physically this makes sense, because the charging mechanisms depend on the grain velocities.

Coming back to the wave dispersion properties [8], we note that the correct definition of the polydisperse dust plasma frequency is through $\omega_{pd}^2 = \xi_{d0}/\varepsilon_0$, determined by the average of the ratio of the charge squared to the mass. The subscript 0 refers to equilibrium values, to be computed from the equilibrium distribution f_{d0}. For the dust Jeans frequency squared $\omega_{Jd}^2 = 4\pi G \rho_{d0}$ the average mass $\overline{m}_d = \rho_{d0}/n_{d0} = \langle m \rangle / \langle 1 \rangle$ enters. This tallies with previous attempts to study dust mass distributions [6,7].

The dust-acoustic dispersion law is [8]

$$\omega^4 + \omega^2(\omega_{Jd}^2 - k^2 c_{da}^2) + k^2(\overline{c}_{da}^2 - c_{da}^2)\omega_{Jd}^2 = 0, \tag{10}$$

where electron and ion inertia are neglected and λ_D is the effective plasma Debye length [3]. There are two dust-acoustic velocities $c_{da} = \omega_{pd}\lambda_D$ and $\overline{c}_{da} = \overline{\omega}_{pd}\lambda_D$, corresponding to the two dust plasma frequencies. The latter occurs in problems involving self-gravitation, is defined through $\overline{\omega}_{pd}^2 = \sigma_{d0}^2/\varepsilon_0 \rho_{d0}$ and is computed from the average charge and mass densities, or equivalently from the number density $n_{d0} = \langle 1 \rangle$, the average dust charge $\overline{q}_d = \langle q \rangle / \langle 1 \rangle$ and the average dust mass $\overline{m}_d = \langle m \rangle / \langle 1 \rangle$. Also, $\overline{\omega}_{pd}$ only reduces to ω_{pd} provided all dust grains have the same charge-to-mass ratio. Moreover, since $\xi_{d0} \geqslant \sigma_{d0}^2/\rho_{d0}$ [8], it follows that in general $\overline{\omega}_{pd}^2 \leqslant \omega_{pd}^2$, and similarly for the dust-acoustic velocities $\overline{c}_{da}^2 \leqslant c_{da}^2$.

For monodisperse dust, or for dust grains all having the same charge-to-mass ratios, the last term in Eq. (10) vanishes and we recover the dust-acoustic Jeans mode [6,13]. Since one of the roots ω^2 goes to zero when $\overline{c}_{da} \rightarrow c_{da}$, that mode is called the dust distribution mode [6,8].

III. CONTINUOUS SIZE DISTRIBUTIONS

We now adapt these definitions and results to size distributions, rather than treating charge and mass as uncorrelated phase space variables. This is particularly relevant when the orbit-motion-limited (OML) charging model [3,4] is used, where the dust is charged by electron and ion currents to the grains, and in a given plasma environment the charge is proportional to the size a. Furthermore, for grains of similar composition the mass varies with a^3.

A redefinition of the distribution function f_d to depend on a rather than on q and m separately yields

$$\xi_d = \int g_d(a) \frac{q(a)^2}{m(a)} da, \tag{11}$$

with analogous expressions for σ_d and ρ_d. Taking $q(a) \propto a$ and $m(a) \propto a^3$ allows a connection with earlier attempts to describe charged dust grains with sizes in a continuous but limited size range [7,14–16].

Now we turn to an equivalent monodisperse description, and try to see how an average dust grain might be defined. To fix the ideas, we assume that all grains are made up of the same material, with mass density ρ_{mass}, and charged in the same plasma and radiation environment to a surface potential V. Then a simple link exists between size, charge and mass, $q(a) = Qa$ and $m(a) = Ma^3$, if the coefficients of proportionality in the OML charging model are denoted by $Q = 4\pi\varepsilon_0 V$ and $M = 4\pi\rho_{\text{mass}}/3$, respectively [3,4]. In our notation with the averages, one has that

$$\sigma_{d0} = Q\langle a \rangle, \ \rho_{d0} = M\langle a^3 \rangle, \ \xi_{d0} = \frac{Q^2}{M}\left\langle \frac{1}{a} \right\rangle. \tag{12}$$

Hence $\overline{a} = \langle a \rangle / \langle 1 \rangle$ gives the average size of a dust grain, and this conserves overall dust charge density [7]. For comparison with monodisperse grains it seems logical to define the average grain as having an average size. However, the mass of a grain of average size $\widetilde{m}_d = M\overline{a}^3 = M\langle a \rangle^3/\langle 1 \rangle^3$, is quite different from the average grain mass $\overline{m}_d = M\langle a^3 \rangle / \langle 1 \rangle$.

Turning to the dust plasma frequencies, there now occurs a third possible definition, denoted by $\widetilde{\omega}_{pd}$, namely, that corresponding to a monodisperse description at the average size. Leaving out for all three definitions the common factors $Q^2/\varepsilon_0 M$, we have $\omega_{pd}^2 \propto \langle a^{-1} \rangle$, $\overline{\omega}_{pd}^2 \propto \langle a \rangle^2 / \langle a^3 \rangle$, and $\widetilde{\omega}_{pd}^2 \propto \langle 1 \rangle^2 / \langle a \rangle$. From

$$\omega_{pd}^2 - \widetilde{\omega}_{pd}^2 \propto \langle a \rangle \langle a^{-1} \rangle - \langle 1 \rangle^2 = \frac{1}{2}\int g_d(a) g_d(a') \frac{(a-a')^2}{aa'} da\, da' \geqslant 0,$$

$$\widetilde{\omega}_{pd}^2 - \overline{\omega}_{pd}^2 \propto \langle 1 \rangle^2 \langle a^3 \rangle - \langle a \rangle^3 = \frac{1}{6}\int g_d(a) g_d(a') g_d(a'') \times (a+a'+a'')[(a-a')^2 + (a-a'')^2 + (a'-a'')^2] da\, da'\, da'' \geqslant 0, \tag{13}$$

the ordering between the three definitions of the dust plasma frequencies follows as

$$\overline{\omega}_{pd}^2 \leqslant \widetilde{\omega}_{pd}^2 \leqslant \omega_{pd}^2 \tag{14}$$

and the monodisperse average $\widetilde{\omega}_{pd}$ lies between the two others. Hence, using a monodisperse description at the average size tends to underestimate the mass effects of the distribution, because $\overline{m}_d - \widetilde{m}_d \propto \langle 1 \rangle^2 \langle a^3 \rangle - \langle a \rangle^3 \geqslant 0$. The same ordering occurs for the corresponding dust-acoustic velocities, with $\widetilde{c}_{da} = \widetilde{\omega}_{pd} \lambda_D$ being smaller than the correct average c_{da} over the dust distributions. Finally, there are analogous repercussions on the Jeans frequency defined with $\widetilde{m}_d$ rather than with $\overline{m}_d$, so that $\widetilde{\omega}_{Jd}^2 = 4\pi G n_{d0} \widetilde{m}_d \leqslant \omega_{Jd}^2$.

For several space dust distributions there occurs a power-law density decrease with size, of the form $g_d(a) = Ca^{-\mu}$, where μ is positive and fairly large, of order 3 to 7. Distributions of this sort have been observed in planetary dusty plasmas, with power-law indices $\mu = 4.6$ for the F ring of Saturn [17], and $\mu = 6$ [18] or $\mu = 7$ [19] for the G ring. The integrals over size can then be worked out explicitly,

$$\mathcal{I}_\ell = \langle a^\ell \rangle = \int_{a_{\min}}^{a_{\max}} g_d(a) a^\ell\, da = C \frac{a_{\min}^{\ell+1-\mu} - a_{\max}^{\ell+1-\mu}}{\mu - \ell - 1}. \tag{15}$$

For values of $\mu > \ell + 1$, viz., for rather steep power-law decreases and provided the ratio $a_{\max}/a_{\min}$ is sufficiently large, the integrals can be approximated as

$$\mathcal{I}_\ell \simeq C \frac{a_{\min}^{\ell+1-\mu}}{\mu - \ell - 1}. \tag{16}$$

It is then the minimum size of the dust grains that plays a crucial role, but unfortunately this quantity is difficult to ascertain observationally, since it could be well below the sensitivity of the detectors. Thus we obtain

$$\overline{\omega}_{pd}^2 \propto \frac{\mathcal{I}_1^2}{\mathcal{I}_3} \simeq \frac{C(\mu-4)}{(\mu-2)^2} a_{\min}^{-\mu},$$

$$\tilde{\omega}_{pd}^2 \propto \frac{\mathcal{I}_0^2}{\mathcal{I}_1} \simeq \frac{C(\mu-2)}{(\mu-1)^2} a_{\min}^{-\mu},$$

$$\omega_{pd}^2 \propto \mathcal{I}_{-1} \simeq \frac{C}{\mu} a_{\min}^{-\mu}, \tag{17}$$

and there are some corrections when μ goes through a critical value or when $\mu<4$, as discussed elsewhere [7]. It also follows that the ratios

$$\frac{\tilde{\omega}_{pd}^2}{\omega_{pd}^2} = \frac{\mathcal{I}_0^2}{\mathcal{I}_1 \mathcal{I}_{-1}} \simeq \frac{\mu(\mu-2)}{(\mu-1)^2},$$

$$\frac{\overline{\omega}_{pd}^2}{\omega_{pd}^2} = \frac{\mathcal{I}_1^2}{\mathcal{I}_3 \mathcal{I}_{-1}} \simeq \frac{\mu(\mu-4)}{(\mu-2)^2} \tag{18}$$

depend upon the power-law index, but only very weakly on the limits of the distribution range. When $\mu>4$ but close to 4, these ratios can drop significantly below 1, the limiting value for large μ. While the second ratio only crops up when self-gravitation is present, it is the first that would have wider usage. However, for the μ values observed in planetary rings a monodisperse description would give quite acceptable values for the plasma frequency and the dust-acoustic velocity.

IV. CONCLUSIONS

For a continuous distribution of charged grains with individual characteristics, three definitions of the dust plasma frequency, and hence of the dust-acoustic velocity, occur in a natural way. The dispersion law for dust-acoustic modes shows that one definition is based on the average over all dust grains of the charge-squared-to-mass ratio, whereas a second one uses the average charge and the average mass. The difference between the two gives rise to a dust distribution mode.

A third option is to replace dust grains in a range of sizes by an average monodisperse description, based on grains of an average radius that conserves overall charge density. The dust plasma frequency thus obtained is intermediate between the two others, indicating that assuming a monodisperse description at this average size underestimates the mass effects of the distribution.

In passing, we have discussed the possibility of describing the dust charge (and mass) distributions independently of their momenta, and shown that this is tantamount to saying that all dust grains have the same charge-to-mass ratio, if this separability is to hold for all q and m moments. Hence, one cannot really separate the velocity or momentum aspects of the charge and mass distribution, a reflection of the fact that the charging mechanisms depend on the velocities of the grains.

Our results are of direct interest to various astrophysical dusty plasmas, where dust grains occur naturally in a whole range of sizes, and hence also in a range of charges, depending on the charging mechanisms, as illustrated here for the case of planetary rings.

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